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The Abelian distribution

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We define the Abelian distribution and study its basic properties. Abelian distributions arise in the context of neural modeling and describe the size of neural avalanches in fully-connected integrate-and-fire models of self-organized criticality in neural systems.

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1. Introduction

The distribution that we will discuss here was called the Abelian [8] because its analysis involves a number of identities that resemble the Abel identity $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x(x-iz)^{i-1}(y+iz)^{n-i}$ [13]. This distribution appeared in 2002 in the study of a fully connected neural network [5] as a distribution of sizes of *avalanches* of neural activity. Apart from [8], so far no systematic and accessible study of the distribution has been published. The related results that were reported in the context of Cayley's theorem [4] are also based on [8]. Here we will discuss the basic properties of this probability mass distribution and describe its importance for the applications in theoretical physics and biology.

2. Definition

Definition 2.1. Let $N \in \mathbb{N}$, $\alpha \in (0,1)$. The Abelian distribution is defined for $0 \leq L \leq N$ by

$$P_{\alpha,N}(L) = C_{\alpha,N} \binom{N}{L} \left(L\frac{\alpha}{N}\right)^{L-1} \left(1 - L\frac{\alpha}{N}\right)^{N-L-1}, \tag{2.1}$$

where

$$C_{\alpha,N} = \frac{1-\alpha}{N-(N-1)\alpha} \tag{2.2}$$

is a normalizing constant.

As $P_{\alpha,N}(0) = 0$, we will in the following often assume that $L > 0$.

Lemma 2.1. *The Abelian distribution defined by* (2.1), (2.2) *is a probability distribution.*

Proof. We have to show that

$$\sum_{L=1}^{N} C_{\alpha,N} \binom{N}{L} \left(L\frac{\alpha}{N}\right)^{L-1} \left(1 - L\frac{\alpha}{N}\right)^{N-L-1} = 1.$$

Introducing a new variable $x = \alpha/N$, we get

$$\sum_{L=1}^{N} \binom{N}{L} (Lx)^{L-1} (1-Lx)^{N-L-1} = \frac{1}{C_{\alpha,N}},$$

which is equivalent to

$$\sum_{L=1}^{N-1} \binom{N}{L} (Lx)^{L-1} (1-Lx)^{N-L-1} = \frac{1}{C_{\alpha,N}} - \frac{(Nx)^{N-1}}{1-Nx}. \tag{2.3}$$

We can expand the sum on the left of (2.3) and obtain

$$\sum_{L=1}^{N-1} \binom{N}{L} (Lx)^{L-1} \sum_{m=0}^{N-L-1} (-1)^m \binom{N-L-1}{m} (Lx)^m. \tag{2.4}$$

Introducing $k = L$, we can rewrite the sum in the previous expression as a polynomial in x

$$\sum_{i=0}^{N-2} x^i \sum_{k=0}^{i} (-1)^{i-k} \binom{N}{k} \binom{N-k-1}{i-k} (k)^i = \sum_{i=0}^{N-2} P_i(N) x^i,$$

where $P_i(N)$ is a polynomial in N of degree i. For every N we have $P_0(N) = 1$. Consider now $i > 0$. To identify uniquely the polynomial $P_i(N)$ it is sufficient to find its values in $i+1$ different points that we select to be $N = 1, \ldots, i+1$. As

$\binom{N-1}{k} = 0$ for $k > N-1$, we also have $\binom{N-k-2}{i-k} = 0$ for $N < i+2$ for any $k < N-1$. Hence,

$$P_i(N) = (-1)^{i-k}\binom{N}{N}\binom{-1}{i-k}N^i = N^i \quad \text{for } N = 1,\dots,i+1 \text{ and } i > 0.$$

This means that $P_i(N) = N^{i-1}$ for any N and $i > 0$. Therefore the left of (2.4) is

$$1 + \sum_{i=1}^{N-2} x^i N^{i-1} = 1 + x\frac{1-(Nx)^{N-2}}{1-Nx}. \tag{2.5}$$

Inserting (2.5) and (2.2) into (2.3) we now have to show that

$$1 + x\frac{1-(Nx)^{N-2}}{1-Nx} = \frac{N-(N-1)\alpha}{N(1-\alpha)} - \frac{(Nx)^{N-1}}{N(1-Nx)},$$

which holds for any N and $\alpha < 1$. □

The authors of [4] mention that the theorem can also be proved by using a generalized binomial theorem.

An Abelian-distributed probability mass function is shown in Fig. 1 for several values of the parameter α. For small values of parameter $\alpha < 0.9$ distribution is monotone and is dominated by approximately exponential decay, for $\alpha \lesssim 1$ distribution is non-monotonous. For some small interval of parameter values $\alpha \approx 0.9$ the distribution closely resembles a power-law (with exponential cutoff at large L), see the double logarithmic plot in the inset. If a sample of data-points of size 10^5 is drawn from this distribution, the hypothesis of an underlying power-law distribution cannot be rejected [8].

The shape of the distribution varies in a similar way for all N, although the non-monotonous regime is present only for $\alpha \in (\alpha_c(N), 1)$ where the value of $\alpha_c(N)$ has been numerically found to behave roughly as $1 - \frac{1}{\sqrt{N}}$.

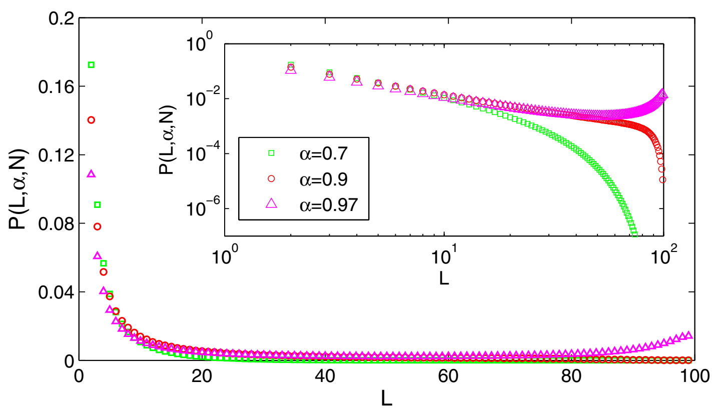

Fig. 1. A probability mass function obeying the Abelian distribution for $N = 100$ and several values of the parameter α. Scales are linear (main figure) and double logarithmic (inset).

3. Expected Value

We will now consider the moments of the Abelian distribution.

Theorem 3.1. *Suppose ξ has an Abelian distribution with parameters α and N, then*

$$E\xi = \frac{N}{N-(N-1)\alpha}.$$

Proof. From (2.1) and Lemma 2.1 we have

$$E\xi = \sum_{L=1}^{N} L^{L-1}\binom{N-1}{L-1}\left(\frac{\alpha}{N}\right)^{L-1}\left(1-L\frac{\alpha}{N}\right)^{N-L-1}\frac{N(1-\alpha)}{N-(N-1)\alpha}.$$

We have to prove that

$$\sum_{L=1}^{N} L^{L-1}\binom{N-1}{L-1}\left(\frac{\alpha}{N}\right)^{L-1}\left(1-L\frac{\alpha}{N}\right)^{N-L-1} = \frac{1}{1-\alpha}.$$

Using again $x=\alpha/N$ we can rewrite this equation as

$$\sum_{L=1}^{N}\binom{N-1}{L-1}(Lx)^{L-1}(1-Lx)^{N-L-1} = \frac{1}{1-Nx}. \tag{3.1}$$

Transforming the sum in (3.1) we obtain

$$\sum_{L=1}^{N-1}\binom{N-1}{L-1}(Lx)^{L-1}(1-Lx)^{N-L-1} + (Nx)^{N-1}(1-Nx)^{-1} = \frac{1}{1-Nx},$$

which is equivalent to

$$\sum_{L=1}^{N-1}\binom{N-1}{L-1}(Lx)^{L-1}(1-Lx)^{N-L-1} = \sum_{i=0}^{N-2}(Nx)^{i}. \tag{3.2}$$

Both sides of Eq. (3.2) are polynomials in x of degree $N-2$. Hence in order to prove that Eq. (3.2) is an identity it is sufficient to show that the coefficients of x^i on both sides are equal for every i. In other words, we have to show that

$$\sum_{k=0}^{i}(-1)^{i-k}\binom{N-1}{k}\binom{N-k-2}{i-k}(k+1)^{i} = N^{i}. \tag{3.3}$$

Again, both sides of (3.3) are polynomials of N of degree i. It is sufficient to prove that both sides of (3.3) are equal for $i+1$ different points. We can select these points to be $N=1,\dots,i+1$.

Obviously, if $k>N-1$, then $\binom{N-1}{k}=0$, but also $\binom{N-k-2}{i-k}=0$ for $k<N-1$ because $N<i+2$. Hence the only nonzero item of the sum is the one corresponding to $k=N-1$, in this case we have

$$(-1)^{i-k}\binom{N-1}{N-1}\binom{-1}{i-k}N^{i} = N^{i}.$$

□

4. Motivation

Power-law distributions have been studied for a long time, the most prominent example being the Gutenberg–Richter law which describes the energy distribution in earthquakes [6]. Other examples [1] include forest fires, migratory patterns, infectious diseases, solar flares, sandpiles [2] and neural activity dynamics [5, 3, 10, 11]. Some of these examples can be related to critical branching processes [7] which are known to produce power-law event distributions [12]. The relation between power-laws and branching processes usually requires a limit of large system size [9] which is, however, not relevant when a comparison to numerical computations or mesoscopic experiments is desired. Nevertheless, the Abelian distribution converges to a power-law with exponent $\gamma = -\frac{3}{2}$ (asymptotically for large event sizes $L \to \infty$ or as an event density) in the exchangeable limits $N \to \infty$ and $\alpha \to 1$. The distribution is well matched by a power-law for small L. Criticality being defined as the divergence of certain physical quantities (such as the mean event size) cannot occur in finite systems. Therefore, it is tempting to use the Abelian distribution to define an analog of criticality also for finite systems. Depending on the parameters, the Abelian distribution has monotonic or non-monotonic behavior, the latter being characterized by a relative dominance of events with a size near the size of the system. The two behaviors, the sub- and the supercritical regime are separated by a "critical" distribution, which is, however, unambiguously defined in terms of a power-law only for large systems. Avoiding the dependence of the critical parameters on the sample size that may arise when using a test (e.g., Kolmogorov–Smirnov) in order to determine the likelihood of criticality, we propose instead to define criticality by qualitative criteria implied by the local similarity to a power-law. Consider the set $A(N)$ of parameters α for which the expression $\frac{d^2 \log P_{\alpha,N}(L)}{d(\log L)^2}$ changes sign between L and $L+1$ for some $L \in \{2, \ldots, N-1\}$. Expecting $A(N)$ to contract into $\{1\}$ for $N \to \infty$, we can define $A(N)$ as the critical region for a finite system. Another possibility is to define a single critical value α_{c} as a $\sup_{\alpha<1}\{\alpha : \frac{d \log P_{\alpha,N}(L)}{dL} < 0, \forall\, L < N\}$. This definition uses the property of the critical state to remain between strictly monotonous and non-monotonous regimes. For all our numerical evaluations we found $\alpha_{\mathrm{c}} \in A(N)$. Thus, the Abelian distribution is one of the few cases where the emergence of criticality in an infinite system can be studied explicitly as a limit of finite systems which enables a direct comparison with numerical computations or mesoscopic experiments.

The Abelian distribution has been studied mainly in the context of neural avalanche dynamics [5, 8], where it not only turned out be successful in predicting an experimental result from neuroscience [3], but also allowed for an explicit and exact study of finite size effects. It is interpreted in this context as the conditional probability of $L-1$ other neurons being activated given that one neuron just became spontaneously active, thus forming an avalanche of L neural action potentials. From Theorem 3.1 follows that the expectation exists also in the limit of large N if $\alpha < 1$ as required by Definition 2.1. Correspondingly, in the neural system, a single nonterminating avalanche is observed at $\alpha = 1$.

The application of the Abelian distribution as an event size distribution may require an appropriate definition of events. Although neurons produce quasi-discrete action potentials, in the experiment [3] events have been defined by threshold crossings, where an invariance of the distribution of the choice of the threshold is required for justification. In other time series, events can be defined in a similar way. While the parameter N usually has a natural interpretation as the size of the system, e.g., financial time series, its meaning is less obvious. If N can be found by maximum likelihood, it can be interpreted as an effective system size. The parameter α describes in all cases the strength of the interaction between the elements in the system. If the elements are not all connected or if the system is heterogeneous, it seems reasonable to use, respectively, connectivity-rescaled parameters or an average interaction strength to determine estimates of this parameter.

5. Open Questions

A large number of questions related to the Abelian distribution are left for future investigation. Most important among them are the higher moments, characteristic function, stability and properties related to parameter estimation. Especially interesting for the application to critical system would be a scaling law for the critical value α_c and relation between different possibilities to define criticality for finite systems.

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