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Avalanche dynamics

Manfred Denker
Pennsylvania State University,
McAllister Room 230, State College, PA 16802, USA
denker@math.psu.edu

Anna Levina
Max Planck Institute for Mathematics in Sciences,
Inselstrasse 22, 04103 Leipzig,
Bernstein Center for Computational Neuroscience,
Göttingen Am Fassberg 12, 37077 Göttingen, Germany
anna@nld.ds.mpg.de

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The avalanche transformation as a model for avalanches in neural dynamics was introduced in [8] in 2008. Here we discuss this transformation in terms of group actions, random dynamics and skew products with a finite invariant measure. The results are based on [8]. Some open problems are mentioned.

Keywords: Random dynamical system; group action; rectangle exchange transformation; ergodicity; topological transitivity.

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1. Introduction

Theoretical modeling of the brain processes aims at understanding experimental results, finding general rules governing the brain dynamics, and making predictions that can be tested in further experiments. That is why theories proposing simple unifying mechanisms attract a lot of attention. One of the recent ideas claims that the brain operates near the critical state which is reached and maintained by self-regulatory dynamics. This concept comes down to the ideas of Per Bak and co-authors in [2], who called it self-organized criticality (SOC) and hypothesized that it can explain a multitude of natural phenomena [1]. It describes systems that have a critical point as an attractor. In such systems the spatial and temporal scale invariance is observed. The experiments of Beggs and Plenz [3] demonstrated that similar behavior is present in the recordings from cortical slices where the cascades of activity termed neuronal avalanches with power law statistics were found. One of the early studies in [7] predicted the results obtained later in the experiments and proposed the model that can be translated into the strict mathematical formalism. Based on [8] we discuss here the properties of the avalanche transformation derived from this model.

Let us first introduce the neuro-physical model underlining the present study. It is composed of the set of connected threshold units representing neurons. Every unit is described by the single variable representing its membrane potential that is the difference in electric potential between the interior and the exterior of a neuron. Each unit integrates inputs coming from the outside of the system and via linear connections to other units. As soon as a unit reaches the threshold, it is reset by subtracting the threshold value and all units connected to it receive an input. For simplicity we consider fully-connected networks. The vector of all membrane potentials and its transformation will be the main object of the following study. Without external input the system stays unchanged. After the external input brings one of the units above the threshold, a cascade of activity propagation resembling neuronal avalanche begins. It was found in [7] that depending on the connection strength, distribution of the avalanche size can vary and for specific parameter value it resembles the critical distribution found in the recordings. The analytical form of the avalanche size distribution in this model was obtained and studied in some details in [7, 9]. It was also shown in [6] to be related to Cayley's theorem.

Despite the great interest to self-organized criticality among physicists, up to date there are only a few mathematical studies related to the phenomenon of self-organized criticality. One of the older SOC models, the Zhang model, was studied by Blanchard *et al.* who have shown that it can be represented by hyperbolic dynamical systems with singularities [4] and later analyzed the transport dynamics in the system and related it to the Lyapunov spectrum.

The avalanche transformation, the object of the present work, was introduced by one of the authors [8]. We recall the definition in Sec. 2. The main objective in this note is to show that it may be viewed as a random dynamical system, a measure preserving finitely generated group action, and a skew product transformation. This requires to prove the existence of invariant measures and the invertibility of the maps defining the random transformation. The results presented below are mostly obtained in [8], here we add some remarks, extensions and give several new adapted proofs. Open questions are discussed, in particular the ergodicity conjecture for the avalanche transformation.

2. Random Dynamical Systems

Let $\mathcal{A} = \{1, \ldots, N\}$, where N is a positive integer and let $\alpha, \delta \in (0, 1)$ where N, α and δ are fixed constants. Each $a \in \mathcal{A}$ acts by a transformation $T_a : [0, 1]^N \to [0, 1]^N$ on the N-dimensional unit cube and will be defined first.

Define an auxiliary map

$$F : \mathbb{R}_+^N \to \mathbb{R}_+^N$$

by $F(\mathbf{x}) = \mathbf{y}$ with

$$y_i = x_i - \mathbb{I}_{[1,\infty)}(x_i) + \alpha \sum_{j=1}^{N} \mathbb{I}_{[1,\infty)}(x_j) \quad 1 \le i \le N,$$

where $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$. Here and in the sequel, for $\mathbf{x} \in \mathbb{R}_+^N$ its coordinates are denoted by lower subscripts, so $\mathbf{x} = (x_1, \ldots, x_N)$. We call the value of the coordinate i the real number x_i. We also let $\mathbb{I}_C$ denote the indicator function of the set C.

Lemma 2.1. *If $N\alpha < 1$, then every point $\mathbf{x} \in \mathbb{R}_+^N$ is eventually mapped by F to a point in $[0,1)^N$. Points in $[0,1)^N$ are fixed points of F.*

Proof. ([8]) Let $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$. Then for $\mathbf{y} = (y_1, \ldots, y_N) = F(\mathbf{x})$ one easily finds

$$\begin{aligned} \sum_{i=1}^{N} y_i &= \sum_{i=1}^{N} \left[x_i - \mathbb{I}_{[1,\infty)}(x_i) + \alpha \sum_{j=1}^{N} \mathbb{I}_{[1,\infty)}(x_j) \right] \\ &= \sum_{i=1}^{N} x_i - \sum_{j=1}^{N} (1 - N\alpha) \mathbb{I}_{[1,\infty)}(x_j) \le \sum_{i=1}^{N} x_i. \end{aligned} \tag{2.1}$$

It follows that the sequence $s_n = \sum_{i=1}^{N} F^n(\mathbf{x})_i$ is decreasing, and strictly decreasing by at least the amount of $1 - N\alpha$ if at least one of the coordinates of $F^n(\mathbf{x})$ has value ≥ 1. Therefore, there is n such that $s_n = s_{n+1}$ for the first time. By definition of the map F any point $\mathbf{x} \in [0,1)^N$ is a fixed point. □

Lemma 2.1 shows that the following transformation is well defined.

Let $\mathbf{A} \subset \mathcal{A}^{\mathbb{N}}$ be a subshift of the infinite product space over the finite set $\mathcal{A}$. Then $\mathbf{A}$ is a compact space in the product topology. Let μ denote a shift invariant probability measure on the Borel σ-field $\mathcal{B}(\mathbf{A})$ with support contained in $\mathbf{A}$, where the shift map $\tau : \mathbf{A} \to \mathbf{A}$ defined as $\tau(\mathbf{a}) = \mathbf{b}$ with

$$b_n = a_{n+1} \quad (n \in \mathbb{N}), \quad \mathbf{a} = (a_n)_{n \in \mathbb{N}}, \quad \mathbf{b} = (b_n)_{n \in \mathbb{N}} \in \mathbf{A},$$

is a continuous map.

Our model will be a random dynamical system over the probability preserving dynamical system $(\mathbf{A}, \mathcal{B}(\mathbf{A}), \mu, \tau)$. Each $a \in \mathcal{A}$ acts by a transformation $T_a : [0,1]^N \to [0,1]^N$ on the unit cube and is defined as follows:

Definition 2.1. ([8]) Let $N \in \mathbb{N}$, $0 < \alpha < \frac{1}{N}$ and $\delta > 0$. The avalanche transformation is a random transformation $(\mathbf{A}, \mathcal{B}(\mathbf{A}), \mu, \tau, \mathcal{T})$ where $\mathcal{T} = \{T_a : [0,1]^N \to [0,1]^N : a \in \mathcal{A}\}$ is defined by

$$T_a(\mathbf{x}) = F^{D(a,\mathbf{x})}(S_a(\mathbf{x})),$$

where

$$S_a : \mathbb{R}_+^N \to \mathbb{R}_+^N,$$

$$S_a(\mathbf{x}) = S_a((x_1, \ldots, x_N)) = (x_1, \ldots, x_{a-1}, x_a + \delta, x_{a+1}, \ldots, x_N)$$

and

$$D(a, \mathbf{x}) = \min\{m : F^m(S_a(\mathbf{x})) \in [0, 1)^N\}. \tag{2.2}$$

Definition 2.2. The function $D : \mathcal{A} \times \mathbb{R}_+^N \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ defined by (2.2) is called the duration (of the avalanche), while

$$\xi(a, \mathbf{x}) = \sum_{n=0}^{D(a,\mathbf{x})} \sum_{i=1}^{N} \mathbb{I}_{[1,\infty)}(F^n(S_a(\mathbf{x}))_i) \tag{2.3}$$

is called the size (of the avalanche).

Using this definition we obtain a simple formula to describe the transformation.

Proposition 2.1. *Let $a \in \mathcal{A}$ and $\mathbf{x} = (x_1, \ldots, x_N) \in [0, 1)^N$. Then*

$$T_a(\mathbf{x}) = (x_i + \xi(a, \mathbf{x})\alpha + \delta\delta_{i,a})_{i=1}^N \quad mod\ 1,$$

where $\delta_{l,k}$ denotes the Kronecker symbol, 1 if $l = k$ and 0 otherwise.

We collect some elementary properties of the avalanche size in the next proposition.

Proposition 2.2. *Let $\alpha N + \delta < 1$. Then for $\mathbf{x} \in [0, 1)^N$ and $a \in \mathcal{A}$*

$$\max_{1 \le i \le N} |\{0 \le j \le D(a, \mathbf{x}) : F^j(S_a(\mathbf{x}))_i \ge 1\}| \le 1 \tag{2.4}$$

and

$$\xi(a, \mathbf{x}) = \max\{l : |\{1 \le j \le N : S_a(\mathbf{x})_j \ge 1 - (l - 1)\alpha\}| \ge l\}. \tag{2.5}$$

Proof. We first prove (2.4). Assume the equation does not hold. Then there are coordinates i for which there are at least two iterates $1 \le j \le D(a, \mathbf{x})$ with $F^j(S_a(\mathbf{x}))_i \ge 1$. For such i let $j_1(i) < j_2(i)$ denote the two smallest iterates j and choose m such that

$$n := j_2(m) = \min\{j_2(i) : i\ j_2(i) \text{ exists}\}.$$

Then we obtain

$$\max_{1 \le \ell \le N} \sum_{k=1}^{n-1} \mathbb{I}_{[1,\infty)}(F^k(S_a(\mathbf{x}))_\ell) \le 1$$

and

$$\sum_{k=1}^{n} \mathbb{I}_{[1,\infty)}(F^k(S_a(\mathbf{x}))_m) \ge 2.$$

Therefore

$$1 \le F^n(S_a(\mathbf{x}))_m \le x_m + N\alpha + \delta - 1 < x_m < 1$$

yields a contradiction.

Next we show (2.5). By (2.4) and the definition of ξ there are exactly $\xi(a, \mathbf{x})$ coordinates whose orbit exceeds the level 1 (and only once). For these coordinates i it means that $S_a(\mathbf{x})_i \ge 1 - (\xi(a, \mathbf{x}) - 1)\alpha$. Thus

$$\xi(a, \mathbf{x}) \le M := \max\{l : |\{1 \le j \le N : S_a(\mathbf{x})_j \ge 1 - (l-1)\alpha\}| \ge l\}.$$

In particular, $\xi(a, \mathbf{x}) = 0$ if and only if $S(\mathbf{x})_a < 1$.

Since $F^{D(a,\mathbf{x})}(S_a(\mathbf{x}))$ has all coordinates with values < 1 there are exactly $\xi(a, \mathbf{x})$ coordinates i which are $\ge 1 - (\xi(a, \mathbf{x}) - 1)\alpha$ and all other coordinates satisfy

$$F^{D(a,\mathbf{x})}(S_a(\mathbf{x}))_i = x_i + \xi(a, \mathbf{x})\alpha < 1.$$

Therefore there is no index i so that

$$1 - \xi(a, \mathbf{x})\alpha \le x_i < 1 - (\xi(a, \mathbf{x}) - 1)\alpha$$

and therefore

$$\begin{aligned} &|\{1 \le j \le N : x_j \ge 1 - \xi(a, \mathbf{x})\alpha\}| \\ &\quad = |\{1 \le j \le N : x_j \ge 1 - (\xi(a, \mathbf{x}) - 1)\alpha\}| \\ &\quad = \xi(a, \mathbf{x}). \end{aligned}$$

Thus $M = \xi(a, \mathbf{x})$. □

Remark 2.1. The Markov operator associated to the random dynamical system is

$$Pf(\mathbf{x}) = \sum_{a \in \mathcal{A}} \int_{\mathbf{A}} f(T_a(\mathbf{x})\mathbb{I}_{[a]}(\mathbf{a})\mu(d\mathbf{a}).$$

A probability measure ν on $[0, 1)^N$ is called invariant if $P\nu = \nu$, where

$$\int f dP\nu = \int Pf(\mathbf{x})\nu(d\mathbf{x}).$$

We will show below that there is an invariant measure for P. We conjecture that this invariant measure is ergodic (if α and δ are irrational and rationally indpendent) provided the measure μ on $\mathbf{A}$ is ergodic and is positive on all nonempty open sets. It is also not known what mixing properties this invariant measure enjoys. Since single maps are like rotations, it would not be surprising to find that the invariant measure is unique.

We next collect some elementary facts about the dynamics.

For $l = 1, \ldots, N$ define

$$W_l = \{\mathbf{x} \in [0, 1)^N : |\{1 \le i \le N : x_i < l\alpha\}| \ge l\},$$

and let W be the union of all W_l.

It is easy to see that

$$W_1 = \bigcup_{i=1}^{N} \{(u_1, \ldots, u_N) \in [0,1)^N : \; 0 \le u_i < \alpha\}$$

and

$$W_2 = \bigcup_{i \neq j = 1}^{N} \{(u_1, \ldots, u_N) \in [0,1)^N : \; 0 \le u_i, u_j < 2\alpha\}.$$

For $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$ define the permutation $\pi_{\mathbf{x}} = (\pi_{\mathbf{x}}(1), \ldots, \pi_{\mathbf{x}}(N))$ of $\{1, \ldots, N\}$ by

$$x_{\pi_{\mathbf{x}}(1)} \le x_{\pi_{\mathbf{x}}(2)} \le \cdots \le x_{\pi_{\mathbf{x}}(N)} \quad \text{and} \quad x_{\pi_{\mathbf{x}}(i)} = x_{\pi_{\mathbf{x}}(i+1)} \Rightarrow \pi_{\mathbf{x}}(i) < \pi_{\mathbf{x}}(i+1). \tag{2.6}$$

$\pi_{\mathbf{x}}$ is uniquely determined, so we can define

$$C = \{\mathbf{x} = (x_1, \ldots, x_N) \in [0,1)^N : x_{\pi_{\mathbf{x}}(i)} \ge i\alpha, \; i = 1, \ldots, N\}. \tag{2.7}$$

It is easy to see that

$$C = W^c = [0,1)^N \setminus W.$$

Lemma 2.2. *For every $a \in \mathcal{A}$ we have that*

$$W \cap T_a(C) = \emptyset.$$

Proof. Assume the claim is not true. Then there is $\mathbf{x} \in W$ and a point $\mathbf{y} \in C = [0,1)^N \setminus W$ such that $T_a(\mathbf{y}) = \mathbf{x}$. Assume first that there exist a coordinate of $S_a(\mathbf{y})$ with value larger than or equal to 1. Then

$$\mathbf{x} = F^{D(a,\mathbf{y})}(S_a(\mathbf{y}))$$

by definition. Set $D = D(a, \mathbf{y})$ and let C_j denote the set of coordinates with values exceeding 1 after applying F^{j-1}, $j = 1, \ldots, D$. Then for $i \in C_j$

$$\begin{aligned} z_i &= F^{j-1}(S_a(\mathbf{y}))_i \ge 1 \\ x_i &= F^{D-j+1}(F^{j-1}(S_a(\mathbf{y})))_i \ge (|C_j| + |C_{j+1}| + \cdots + |C_D|)\alpha, \end{aligned} \tag{2.8}$$

and for $i \notin \bigcup_{j=1}^{D} C_j$

$$x_i \ge (|C_1| + |C_2| + \cdots + |C_D|)\alpha = \xi(a, \mathbf{y})\alpha.$$

Since $\mathbf{x} \in W$ there exists $1 \le L \le N$ and coordinates $i_1, \ldots, i_l$ such that $x_{i_l} < L\alpha$ and $l \ge L$. If $L \le \xi(a, \mathbf{y}) = |C_1| + \cdots + |C_D|$, i.e. for some j

$$|C_j| + |C_{j+1}| + \cdots + |C_D| < L \le |C_{j-1}| + |C_j| + \cdots + |C_D|,$$

then only the coordinates $i \in C_j \cup \ldots \cup C_D$ can satisfy $x_i < L\alpha$ by (2.8), but there are less than L of these coordinates, a contradiction. It follows that $L > \xi = \xi(a, \mathbf{y})$.

In this case there are $L - \xi$ coordinates not in $C_1 \cup ... \cup C_D$ which are bounded by $L\alpha$. Then, for these coordinates,

$$0 \leq y_i = x_i - \xi\alpha < (L - \xi)\alpha.$$

This implies that $\mathbf{y} \in W$, a contradiction.

It remains to consider the case $S_a(\mathbf{y}) \in [0,1)^N$. Since then $T_a(\mathbf{y}) = S_a(\mathbf{y})$, it follows that $y_a = x_a - \delta \geq 0$ and $y_i = x_i$ for all other coordinates i. Since $\mathbf{x} \in W$ there is a subset $J \subset \{1, \ldots, N\}$ consisting of at least $|J|$ coordinates of $\mathbf{x}$ with values that are bounded above by $|J|\alpha$. Since $0 \leq y_i \leq x_i$, there are also at least $|J|$ coordinates of $\mathbf{y}$ with values not exceeding $|J|\alpha$. Thus $\mathbf{y} \in W$, again a contradiction.

This shows the claim. □

We say that a point in $\mathbf{x} \in [0,1)^N$ is wandering for the random transformation $(\mathbf{A}, \mathcal{B}(\mathbf{A}), \mu, \tau, \mathcal{T})$ if for every infinite sequence $(a_1, a_2, \ldots) \in \mathbf{A}$ there is an open set U containing $\mathbf{x}$ such that for any $n \geq 1$

$$U \cap T_{a_n} \circ T_{a_{n-1}} \circ \cdots \circ T_{a_1}(U) = \emptyset.$$

A set $Y \subset [0,1)^N$ is called wandering for the random transformation if every point of W is wandering for $\mathcal{T}$.

Theorem 2.1. *Let* $(\mathbf{A}, \mathcal{B}(\mathbf{A}), \mu, \tau, \mathcal{T})$ *be an avalanche transformation with parameters* $N \in \mathbb{N}, 0 < \alpha < \frac{1}{N}$ *and* $\delta > 0$. *Then every point in* W *is wandering.*

Proof. In order to show the theorem, it suffices to show that for every sequence $a_1, a_2, \ldots \in \mathcal{A}^{\mathbb{N}}$ the point $\mathbf{x} \in W$ eventually maps to W^c under some transformation $\widetilde{T}_n := T_{a_n} \circ T_{a_{n-1}} \circ \cdots \circ T_{a_1}$, $n \geq 1$. If so, we can choose a neighborhood $U \subset W$ which maps in finite time into C and is pairwise disjoint until that iteration. Then by Lemma 2.2, $\widetilde{T}_m(U)$ will never intersect U for $m \geq n$.

So let $a_k \in \mathcal{A}$, $k \geq 1$, and $\mathbf{x} \in W$. Let $L = L(\mathbf{x}) \leq N$ be the minimal integer such that the number of coordinates of $\mathbf{x}$ with values below αL is at least L. An application of T_{a_1} when $S_{a_1}(\mathbf{x}) \in [0,1)^N$ yields the point $(x_1, \ldots, x_{a_1-1}, x_{a_1} + \delta, x_{a_1+1}, \ldots, x_N)$. Thus, repeating if necessary, there is a small neighborhood U of $\mathbf{x}$ and $n \geq 1$ such that $\widetilde{T}_k(U)$, $0 \leq k \leq n$, are pairwise disjoint and $S_{a_n}(\widetilde{T}_{n-1}(\mathbf{x}))_{a_n} > 1$. Therefore we may assume that $n = 1$ and $S_{a_1}(\mathbf{x}) \notin [0,1)^N$. Now we can argue as in the proof of Lemma 2.2: Let C_j denote the set of coordinates with values that exceed 1 after the $(j-1)$th iteration of F. Then for each $L' \leq \xi(a, \mathbf{x})$ the number of coordinates of $T_{a_1}(\mathbf{x})$ with values not exceeding $\alpha L'$ is less than L'. For $L' > \xi(a, \mathbf{x})$ let C_0 denote the set of coordinates of $T_{a_1}(\mathbf{x})$ with values that are bounded by $\alpha L'$. This set C_0 splits into two parts, the union of all C_j and those coordinates of $\mathbf{x}$ with values that are bounded by $\alpha(L' - \xi(a, \mathbf{x}))$. Therefore the minimal $L = L(T_{a_1}(\mathbf{x}))$ such that the number of coordinates of $T_{a_1}(\mathbf{x})$ with values that are bounded by αL is at least L is related to $L(\mathbf{x})$ by

$$L(T_{a_1}(\mathbf{x})) = \min\{N, L(\mathbf{x}) + \xi(a, \mathbf{x})\} > L(\mathbf{x}).$$

Iteration clearly shows that the orbit of $\mathbf{x}$ leaves W. □

3. Group Actions

In this section we show that the transformations T_a introduced in Definition 2.1 generate a group action on the subspace $C \subset [0,1]^N$ defined in (2.7). This follows from the next two propositions. It is sometimes more convenient to represent the transformation T_a restricted to C in a more tractable form.

Lemma 3.1. *For $z \in \mathbb{R}_+^N$ let $\pi_{\mathbf{z}}$ denote the permutation defined in* (2.6). *Then for $a \in \mathcal{A}$ there exists a unique $\xi = \xi(\mathbf{x})$ such that*

$$T_a(\mathbf{x}) = \begin{cases} (x_1, \ldots, x_{a-1}, x_a + \delta, x_{a+1}, \ldots, x_N) & \text{if } (a) \\ (x'_1 + \eta_1(\mathbf{x})\alpha, \ldots, x'_{a-1} + \eta_{a-1}(\mathbf{x})\alpha, x'_a + \xi\alpha, & \\ \qquad x'_{a+1} + \eta_{a+1}(\mathbf{x})\alpha, \ldots, x'_N + \eta_N(\mathbf{x})\alpha) & \text{if } (b), \end{cases}$$

where

$$\begin{cases} \text{(a)} \quad x_a < 1 - \delta \\ \text{(b)} \quad x_a \geq 1 - \delta, \ x_{\pi_{S_a(\mathbf{x})}(N-\xi)} < 1 - \xi\alpha \ \text{ and } \ x_{\pi_{S_a(\mathbf{x})}(N-i)} \geq 1 - i\alpha, \quad 1 \leq i < \xi, \end{cases}$$

and where for $j \neq a$

$$\eta_j(\mathbf{x}) = \eta_j(a, \mathbf{x}) = \begin{cases} \xi & \text{if } x_j < 1 - \xi\alpha \\ \xi - l & \text{if } 1 - l\alpha \leq x_j < 1 - (l-1)\alpha, \ \text{for some } 1 \leq l < \xi \end{cases}$$

and

$$x'_j = \begin{cases} x_j + (\xi - \eta_j(\mathbf{x}))\alpha - 1 & \text{if } x_j > 1 - \xi\alpha, \\ x_a + \delta - 1 & \text{if } j = a, \\ x_j & \text{if } x_j < 1 - \xi\alpha. \end{cases}$$

Proof. It is sufficient to note that ξ as required in the lemma satisfies

$$\xi = \xi(a, \mathbf{x}) \quad \text{(cf. (2.3))}. \qquad \square$$

The following two propositions establish that each T_a is invertible. The reason why T_a is invertible is geometrically quite clear and intuitive.

Proposition 3.1. *Let $\delta < \alpha$ and $N\alpha < 1$. Then for every $a \in \mathcal{A}$ the transformation T_a is injective on the set C.*

Proof. Let $\mathbf{x}, \mathbf{y} \in C$ and $T_a(\mathbf{x}) = T_a(\mathbf{y})$. Because of symmetry and since T_a is order preserving for all coordinates $\neq a$, we may assume that $a = N$ and $x_i \leq x_j \bmod 1 - \alpha$ if and only if $y_i \leq y_j \bmod 1 - \alpha$ for $i, j < N$. By Lemma 3.1 it follows immediately that $\mathbf{x} = \mathbf{y}$ in case that both points satisfy condition (a) or (b) respectively. Hence we need to consider the case that $\mathbf{x}$ satisfies condition (a) and

$\mathbf{y}$ condition (b). Then, since $N\alpha + \delta < 1$ we have that

$$\alpha \le x_N + \delta = y'_N + \xi(\mathbf{y})\alpha < \xi(\mathbf{y})\alpha + \delta,$$
$$\alpha \le x_{\pi_{S_a(\mathbf{y})}(N-i)} = y'_{\pi_{S_a(\mathbf{y})}(N-i)} + \xi_i(\mathbf{y})\alpha \qquad 1 \le i < \xi(\mathbf{y}),$$
$$\alpha \le x_{\pi_{S_a(\mathbf{y})}(N-i)} = y_{\pi_{S_a(\mathbf{y})}(N-i)} + \xi(\mathbf{y})\alpha \qquad i \ge \xi(\mathbf{y}).$$

It follows that there are $\xi(\mathbf{y})$ coordinates i of $\mathbf{x}$ with values below $\xi(\mathbf{y})\alpha$, contradicting the definition of C. □

Remark 3.1. The transformation T_a is not injective on $[0,1)^N$ as it can be seen that for $a \in \mathcal{A}$

$$T_a\left(1 - \frac{\alpha\delta}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) = \left(\alpha + \delta\left(1 - \frac{\alpha}{2}\right), \frac{1}{2} + \alpha, \ldots, \frac{1}{2} + \alpha\right)$$

and

$$T_a\left(\alpha\left(1 - \frac{\delta}{2}\right), \frac{1}{2} + \alpha, \ldots, \frac{1}{2} + \alpha\right) = \left(\alpha + \delta\left(1 - \frac{\alpha}{2}\right), \frac{1}{2} + \alpha, \ldots, \frac{1}{2} + \alpha\right).$$

Also, the condition $\delta < \alpha$ is necessary since otherwise take $N = 2$, $\mathbf{x} = (1 - \alpha, \eta)$ and $\mathbf{y} = (1 - 2\alpha, 1 - \alpha + \eta)$ where $\eta > 0$ is sufficiently small. If $\delta > 2\alpha$, then with $a \in \mathcal{A}$

$$T_a(\mathbf{x}) = (\delta, \alpha + \eta)$$

and

$$T_a(\mathbf{y}) = F^2(S_a(\mathbf{y})) = F(\delta - \alpha, 1 + \eta) = (\delta, \alpha + \eta).$$

Proposition 3.2. *Let $\delta < \alpha$ and $N\alpha < 1$. Then for each $a \in \mathcal{A}$ the transformation T_a is surjective on C.*

Proof. Fix $a \in \mathcal{A}$. Define subsets of $C = [0,1)^N \backslash W = \bigcap_{j=1}^N [0,1)^N \backslash W_j$ by

$$C_j = \{\mathbf{z} = (z_1, \ldots, z_N) \in C : |\{1 \le i \le N : i \ne a;\ z_i < \alpha j\}| = j - 1\},$$

$j = 1, \ldots, N$.

Let $z \in C$ and set $C_j(z) = \{1 \le i \le N :\ z_i < \alpha j\}$. Since $z \in C$, we must have by definition that

$$|C_j(z)| \le j - 1 \quad \text{and} \quad C_{j-1}(z) \subset C_j(z).$$

Moreover, let $z \in C_j$ for some $1 \le j \le N$. Then $C_j(z) \backslash C_{j-1}(z) \ne \emptyset$. Define

$$k_l = |C_l(z) \backslash C_{l-1}(z)|.$$

It follows that $|C_j(z) \backslash C_l(z)| = k_j + \cdots + k_{l+1} \ge j - 1 - (l - 1) = j - l$.

Define a transformation $h : C \to C$ as follows. h will be shown to be the inverse of T_a. For $\mathbf{y} = (y_1, \dots, y_N) \in C$ let $\mathbf{y}_a = (y_1, \dots, y_{a-1}, y_a - \delta, y_{a+1}, \dots, y_N)$ and define $j(\mathbf{y})$ by

$$\alpha(j(\mathbf{y}) - 1) \le y_a - \delta < \alpha j(\mathbf{y}), \quad 1 \le j(\mathbf{y}) \le N$$

if $y_a - \delta < \alpha N$ or $j(\mathbf{y}) = 0$ otherwise.

If $j(\mathbf{y}) = 0$ we have $\mathbf{y}_a \in C$ and we set $h(\mathbf{y}) = \mathbf{y}_a$.

If $1 \le j(\mathbf{y}) \le N$ and $|C_{j(\mathbf{y})}(\mathbf{y})| < j(\mathbf{y}) - 1$ then we also put $h(\mathbf{y}) = \mathbf{y}_a$ since $\mathbf{y}_a \in C$ as well.

If $1 \le j(\mathbf{y}) \le N$ and $|C_{j(\mathbf{y})}(\mathbf{y})| = j(\mathbf{y}) - 1$ define

$$h(\mathbf{y}) = (y_i - j(\mathbf{y})\alpha - \delta\delta_{i,a})_{1 \le i \le N} \mod 1.$$

We show first that $h(\mathbf{y}) \in C$.

Let $1 \le l + j(\mathbf{y}) \le N$. Then $C_l(h(\mathbf{y})) = C_{l+j(\mathbf{y})}(\mathbf{y}) \backslash C_{j(\mathbf{y})}(\mathbf{y})$ and

$$|C_l(h(\mathbf{y}))| = |C_{l+j(\mathbf{y})+1}(\mathbf{y})| - |C_{j(\mathbf{y})}(\mathbf{y})| \le l + \zeta(\mathbf{y}) - 1 - \zeta(\mathbf{y}) = l - 1,$$

since $\mathbf{y} \in C$.

If $N < l + j(\mathbf{y})$, the coordinates i with $y_i < \alpha j(\mathbf{y})$ satisfy

$$y_i - j(\mathbf{y})\alpha + 1 \ge (N - j(\mathbf{y}))\alpha,$$

hence

$$|C_l(h(\mathbf{y}))| = |\{1, \dots, a-1, a+1, \dots, N\} \backslash C_{j(\mathbf{y})}(\mathbf{y})| \le N - 1 - (j(\mathbf{y}) - 1) < l.$$

This shows that $y(\mathbf{y}) \in C$.

It remains to show that $T_a(h(\mathbf{y})) = \mathbf{y}$. Note that

$$0 \le h(\mathbf{y})_a + \delta \mod 1 = y_a - j(\mathbf{y})\alpha \le \delta$$

and therefore $D(a, h(\mathbf{y})) \ge 1$. It is left to show that $\xi(a, h(\mathbf{y})) = j(\mathbf{y})$. By definition $h(\mathbf{y})$ has $|C_l(\mathbf{y})|$ coordinates $\neq a$ with values that exceed $1 - j(\mathbf{y})\alpha + (l-1)\alpha$ for $j \le j(\mathbf{y})$. This means that

$$\begin{aligned}|\{1 \le i \le N : i \neq a; h(\mathbf{y})_i \ge 1 - k\alpha\}| &= |C_{j(\mathbf{y})}(\mathbf{y}) \backslash C_{j(\mathbf{y})-k}(\mathbf{y})| \\ &= j(\mathbf{y}) - 1 - |C_{j(\mathbf{y})-k}(\mathbf{y})| \ge j(\mathbf{y}) - 1 - (j(\mathbf{y}) - k - 1) = k.\end{aligned}$$

This implies that $\xi(a, h(\mathbf{y})) \ge j(\mathbf{y})$. □

The two propositions, Propositions 3.1 and 3.2, show that in case of $\delta \le \alpha$ each of the restrictions $g_a : C \to C$, $a \in \mathcal{A}$, defined by

$$g_a(\mathbf{x}) = T_a(\mathbf{x}) = F^{D(a,\mathbf{x})}(S_a(\mathbf{x})) \quad \mathbf{x} \in C$$

is a bijective, piecewise linear isometry on C. Let G denote the group generated by the N maps g_a where $a \in \mathcal{A}$, called the avalanche group. We formulate the result of this section in the following theorem

Theorem 3.1. *Let $\alpha N < 1, \delta < \alpha$ and α, δ be rationally independent. The mappings $g_a : C \to C$ generate a group G of piecewise linear isometries which is invariant with respect to the normalized restricted Lebesgue measure on C.*

Remark 3.2. We conjecture that the group G acts freely on C. For $N = 1$, this is obviously true since $g_a^n(x) = x$ implies $x + k\alpha + n\delta - k = x$ for some $0 \leq k \leq n$ which cannot hold by rational independence. For $N \geq 2$ the group is not Abelian. Rational independence is certainly necessary, but it seems to be difficult to find convenient form for iterations of maps.

Remark 3.3. The avalanche group G defined in Theorem 3.1 has Lebesgue measure as its invariant measure. If $N = 1$, it is unique (since δ and α are rationally independent). Hence it is also ergodic. For $N = 2$ it is also not difficult to show that Lebesgue measure is the unique invariant measure (hence also ergodic). This follows from the fact that

$$\|T_1((x,y)) - T_1((u,v))\| = \|(x-u, y-v)\|$$

or

$$\|T_1^2((x,y)) - T_1^2((u,v))\| = \|(x-u, y-v)\|$$

and hence ergodic averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_1^j((x,y)))$$

have a unique limit for all continuous functions on $[0,1)^2$ (use also the main result in Sec. 4).

For general N we still conjecture that Lebesgue measure is ergodic. Note that Lebesgue measure is not ergodic for the transfornations T_a for fixed $a \in \mathcal{A}$. This can be seen from the fact that the set

$$\{(x_1, \ldots, x_N) \in C : |x_2 - x_3| < \alpha\}$$

is T_1 invariant (T_1 keeps the ordering of the coordinates with values ≥ 2) and does not have full Lebesgue measure.

Remark 3.4. Each transformation g_a is a piecewise linear isometry. In fact, it is not difficult to see that it is linear on certain rectangles which are mapped to rectangles. This raises the question of defining a suitable notion of *rectangle exchange transformation* in analogy to interval exchange transformations. It is certainly desirable to have a characterization of ergodicity and other related notions.

4. Skew Products

The avalanche transformation $(\mathbf{A}, \mathcal{B}(\mathbf{A}), \mu, \tau, \mathcal{T})$ can be restricted to C by Theorem 3.1. Any random transformation defines a skew product in a canonical way.

For convenience, we denote the restrictions g_a of T_a to C also by T_a, hence in this section we consider T_a to be invertible.

Theorem 4.1. *The avalanche transformation defines a skew product transformation* $(\mathbf{A} \times C, \mathcal{B}(\mathbf{A} \times C), T, \mu \times \widetilde{\lambda})$ *by*

$$T(\mathbf{a}, \mathbf{x}) = (\tau(\mathbf{a}), T_{a_1}(\mathbf{x})), \quad \mathbf{a} = (a_1, a_2, \ldots), \mathbf{x} \in C,$$

where $\mathcal{B}(\mathbf{A} \times C)$ *denotes the product* σ*-field and* $\mu \times \widetilde{\lambda}$ *the product measure of a shift-invariant probability measure* μ *on* $\mathbf{A}$ *and the normalized restricted Lebesgue measure* $\widetilde{\lambda}$ *on* C.

Proof. Let $R \subset C$ be a small rectangle which, for each $a \in \mathcal{A}$, is contained in a rectangle where T_a acts linearly. Let $\mathbf{c} = [c_1, \ldots, c_m]$ be any cylinder of length $m \in \mathbb{N}$. Since

$$\begin{aligned} T^{-1}(\mathbf{c} \times R) &= \{((b_n)_{n\in\mathbb{N}}, \mathbf{x}) : T_{b_1}(\mathbf{x}) \in R, b_2 = c_1, \ldots, b_{m+1} = c_m\} \\ &= \bigcup_{i=1}^{N} \{((b_n)_{n\geq 2}, T_i^{-1}(\mathbf{x})) : b_1 = i, b_2 = c_1, \ldots, b_{m+1} = c_m, \mathbf{x} \in R\} \\ &= \bigcup_{i=1}^{N} [i, c_1, \ldots, c_m] \times T_i^{-1}(R), \end{aligned}$$

by Theorem 3.1

$$\begin{aligned} \mu \times \widetilde{\lambda}(T^{-1}(\mathbf{c} \times R)) &= \sum_{i=1}^{N} \mu([i, c_1, \ldots, c_m])\widetilde{\lambda}(T_i^{-1}(R)) \\ &= \sum_{i=1}^{N} \mu([i, c_1, \ldots, c_m])\widetilde{\lambda}(R) = \mu \times \widetilde{\lambda}(\mathbf{a} \times R). \end{aligned}$$ □

We say that the transformation T is topologically transitive if there is a point with a dense orbit (with respect to the product topology on $\mathbf{A} \times C$). This is just the usual definition as for continuous transformations. In fact, it is not hard to show that this holds if and only if there is a dense G_δ set of points with a dense orbit.

Theorem 4.2. *Let* α *and* δ *be irrational and rationally independent*, $0 < \delta < \alpha$ *and* $0 < (N + 1)\alpha < 1$. *If* $\mathbf{A} = \mathcal{A}^{\mathbb{N}}$, *then* T *is topologically transitive.*

The proof uses the following result.

Proposition 4.1. *Let* $\epsilon > 0$. *For any* $x, z \in C$ *there are* M *and a sequence* $a_1, \ldots, a_M \in \mathcal{A}$ *such that for any* $a_{M+1}, \ldots \in \mathcal{A}$

$$|\pi_C(\widetilde{T}^M((a_1, \ldots, a_M, a_{M+1}, \ldots), x)) - z| < \epsilon,$$

where π_C *denotes the projection* $\pi_C : \mathbf{A} \times C \to C$.

Remark 4.1. Given $\epsilon > 0$ it can be shown that the constant M is bounded over all choices of $x, z \in C$. This follows from an inspection of the proof below and boundedness of C. Details can be found in [8] where such a result has been proven first.

Proof. Fix $x, z \in C$ and $\epsilon > 0$.
1. Because of the symmetry of the problem we may assume that the coordinates of z satisfy $z_1 \leq z_2 \leq \cdots \leq z_N$. Since $z \in C$ we must also have that

$$\alpha \leq z_1, 2\alpha \leq z_2, \ldots, N\alpha \leq z_N.$$

Therefore there exist integers $s_i \geq 0$ with

$$i\alpha \leq z_i - s_i\delta < i\alpha + \delta.$$

It follows that it is sufficient to show the claim for $z \in C$ with $i\alpha \leq z_i < i\alpha + \delta$ for $1 \leq i \leq N$.

We reduce further claiming that we may assume that

$$z = (z_1, \ldots, z_N) \in Z := [N\alpha - \delta, N\alpha)^{N-1} \times [N\alpha, N\alpha + \delta).$$

We shall show that for an open and dense set of points $z = (z_1, \ldots, z_N)$ with $i\alpha \leq z_i < i\alpha+\delta$, $i = 1, \ldots, N$, there is a point $\xi \in Z$ and a finite string $b_1, \ldots, b_r \in \mathcal{A}$ satisfying:

(1) For all $\epsilon > 0$ sufficiently small and all $y \in Z$ with $|y - \xi| < \epsilon$ it holds that

$$|\pi_C(T((b_1, \ldots, b_r, a_1, a_2, \ldots), \xi)) - \pi_C(T((b_1, \ldots, b_r, a_1, a_2, \ldots), y))| < \epsilon$$

for all $\mathbf{a} = (a_1, a_2, \ldots) \in \mathbf{A}$.

(2) $\pi_C(T((b_1, \ldots, b_r, a_1, a_2, \ldots), \xi)) = z$ and $\pi_C(T((b_1, \ldots, b_r, a_1, a_2, \ldots), y)) \in \prod_{i=1}^{N}[i\alpha, i\alpha + \delta)$ for all $\mathbf{a} = (a_1, a_2, \ldots) \in \mathbf{A}$.

To begin the construction, fix $k \leq N$. We will construct a map

$$\Sigma_k : Z_k := [k\alpha - \delta, k\alpha)^{k-1} \times \prod_{i=k}^{N}[i\alpha, i\alpha + \delta) \to Z_{k-1}$$

which is a bijection and for which there are finitely many rectangles R decomposing the domain such that Σ_k restricted to R has the form

$$\pi_C(T((\mathbf{ba}), \cdot))$$

for some suitable finite string $\mathbf{b}$ and any $\mathbf{a} \in \mathbf{A}$.

Choose s_N such that $1 - \delta \leq (s_N - 1)\delta + N\alpha < 1$. Then, for a string of N's of length s_N

$$\pi_C(T((N, \ldots, N, a_1, a_2, \ldots), [k\alpha - \delta, k\alpha)^{k-1} \times \prod_{i=k}^{N-1}[i\alpha, i\alpha + \delta)$$
$$\times [N\alpha, 1 - (s_N - 1)\delta)))$$

$$= [(k+1)\alpha - \delta), (k+1)\alpha)^{k-1} \times \prod_{i=k}^{N-1} [(i+1)\alpha, (i+1)\alpha + \delta)$$
$$\times [\alpha + N\alpha + s_N\delta - 1, \alpha + \delta)$$

and for a string of N's of length $s_N - 1$

$$\pi_C\left(T\left((N, \ldots, N, a_1, a_2, \ldots), [k\alpha - \delta, k\alpha)^{k-1} \times \prod_{i=k}^{N-1} [(i+1)\alpha, (i+1)\alpha + \delta)\right.\right.$$
$$\left.\left.\times [1 - (s_N - 1)\delta), N\alpha + \delta\right)\right)$$
$$= [(k+1)\alpha - \delta, (k+1)\alpha)^{k-1} \times \prod_{i=k}^{N-1} [(i+1)\delta, (i+1)\alpha + \delta)$$
$$\times [\alpha, \alpha + N\alpha + s_N\delta - 1).$$

Thus we can map Z_k bijectively onto $[(k+1)\alpha - \delta, (k+1)\alpha)^{k-1} \times \prod_{i=k+1}^{N} [i\alpha, i\alpha + \delta) \times [\alpha, \alpha + \delta)$ where the bijection is piecewise an isometry by maps defining the avalanche transformation. Repeating the contruction $N - k$ additional times we arrive at a map

$$Z_k \to [(N+1)\alpha - \delta, (N+1)\alpha)^{k-1} \times \prod_{i=1}^{N-k+1} [i\alpha, i\alpha + \delta)$$

which is a bijection and is defined by local linear isomorphisms arising from the avalanche transformation.

Since $(N+1)\alpha < 1$ there is $s \geq 1$ such that $1 - \delta \leq (N+1)\alpha + (s-1)\delta < 1$. Let $(t_1, \ldots, t_{k-1}) \in \{s-1, s\}^{k-1}$ and consider the string

$$k-1, \ldots, k-1, k-2, \ldots, k-2, \ldots, 1, \ldots, 1$$

where each symbol i appears t_i times for $1 \leq i \leq k-2$ and $k-1$ appears $t_{k-1} + 1$ times. Define rectangles

$$R(t_1, \ldots, t_{k-1}) = \prod_{i=1}^{k-1} I_i \times \prod_{i=1}^{N-k+1} [i\alpha, i\alpha + \delta),$$

where

$$I_i = \begin{cases} [(N+1)\alpha - \delta, 1 - s\delta) & \text{if } t_i = s, \\ [1 - s\delta, (N+1)\alpha) & \text{if } t_i = s-1. \end{cases}$$

Then by definition

$$\pi_C(T((k-1, \ldots, 1, a_1, a_2, \ldots), R(t_1, \ldots, t_{k-1}))) = \prod_{i-1}^{k-1} J_i \times \prod_{i=k}^{N} [i\alpha, i\alpha + \delta),$$

where for $1 \le i \le k-2$

$$J_i = \begin{cases} [(k-1)\alpha - [1-(N+1)\alpha-(s-1)\delta], (k-1)\alpha) & \text{if } t_i = s, \\ [(k-1)\alpha - \delta, (k-1)\alpha - [1-(N+1)\alpha-(s-1)\delta]) & \text{if } t_i = s-1 \end{cases}$$

and for $i = k-1$

$$J_{k-1} = \begin{cases} [(k-1)\alpha + [(N+1)\alpha + s\delta - 1], (k-1)\alpha + \delta) & \text{if } t_i = s, \\ [(k-1)\alpha, (k-1)\alpha + [(N+1)\alpha + s\delta - 1]) & \text{if } t_i = s-1. \end{cases}$$

It follows that there are 2^N rectangles R decomposing Z_k and 2^N strings $\mathbf{b}_R$ such that for any $\mathbf{a} \in \mathbf{A}$ the sets $\pi_C(T(\mathbf{b}_R\mathbf{a}, R))$ are pairwise disjoint and cover

$$Z_{k-1} = [(k-1)\alpha - \delta, (k-1)\alpha)^{k-2} \times \prod_{i=k-1}^{N} [i\alpha, i\alpha + \delta).$$

Moreover, all these maps are linear isometries.

The concatenation of Σ_N, $\Sigma_{N-1}, \ldots, \Sigma_2$ is as well a bijection from $Z_N = Z$ to $Z_1 = \prod_{i=1}^N [i\alpha, i\alpha + \delta)$. There are finitely many rectangles R such that the bijection agrees with $\pi_C(T(\mathbf{ab}, \cdot))$ on R for a suitable finite string $\mathbf{b}$ and any $\mathbf{a} \in \mathbf{A}$. If z belongs to the image of any interior int R, claim **1** is proven.

2. Let $y = (y_1, \ldots, y_N) \in C$. There are integers $s_i \ge 0$ such that

$$1 - \delta \le y_i + s_i\delta < 1 \quad i = 1, 2, \ldots, N.$$

Thus we find a finite string $b_1, \ldots, b_r$ such that for any $1 \le k \le N$ and all $(a_1, a_2, \ldots) \in \mathbf{A}$

$$\begin{aligned} &\pi_C(T^{r+1}((b_1, \ldots, b_r, k, a_1, a_2, \ldots), y)) \\ &\quad = (y_1 + s_1\delta + N\alpha - 1, \ldots, y_{k-1} + s_{k-1}\delta + N\alpha - 1, y_k + s_k\delta + N\alpha - 1 + \delta, \\ &\qquad y_{k+1} + s_{k+1}\delta + N\alpha - 1, \ldots, y_N + s_N\delta + N\alpha - 1). \end{aligned}$$

Note that the length of the string is bounded by $N\frac{1-\alpha}{\delta} + 1$. Also note that by construction

$$-\delta \le y_i + s_i\delta - 1 < 0 \ (i \ne k) \quad \text{and} \quad 0 \le y_k + (s_k+1)\delta - 1 < \delta.$$

Let $y' = (y'_1, \ldots, y'_N)$ denote this image of y. We also have that

$$y'_i - y'_N = y_i - y_N \mod \delta \quad 1 \le i \le N.$$

This map depends on y and is denoted by Σ_k^y, so $\Sigma_k^y(y) = y'$.

We shall need another transformation applied to y', where $1 \le k \le N$ is the same integer as above. First map y' by the map S_k^r (S_k has been defined in Sec. 2) to reach a point in $\{(\xi_1, \ldots, \xi_N) \in C : \xi_i = y'_i \ (i \ne k);\ 1 - \delta \le \xi_k < 1\}$, then apply S_k once more to obtain a point y'' by the map

$$\pi_C(T((k, \ldots, k, a_1, a_2, \ldots), y')) = y'',$$

where $(a_1, a_2, \ldots) \in \mathbf{A}$ is chosen arbitrarily. Denote this map by $R_k^{y'}$. Note that (since $(N+1)\alpha < 1$)

$$y_i'' = y_i' + \alpha \quad (i \neq k) \quad y_k'' = y_k' + (r+1)\delta + \alpha - 1$$

and

$$y_i'' - y_N'' - y_i' - y_N' = 0 \text{ mod } \delta \quad y_k'' - y_N'' = y_k' - y_N' + \gamma \text{ mod } \delta,$$

where $\gamma = (r+1)\delta - 1$ mod δ.

3. Let $\mathbf{l} = (l_1, \ldots, l_{N-1})$, $l_k \geq 0$. By applying the maps $R_k^{y'} \circ \Sigma_k^y \, l_k$ times successively to x, that is $x^1 = R_1^{x'}(\Sigma_1^x(x))$, $x^2 = R_1^{x^{1\prime}}(\Sigma_1^{x^1}(x^1)), \ldots$, we arrive at a point $x^{\mathbf{l}} = (x_1^{\mathbf{l}}, \ldots, x_N^{\mathbf{l}})$ in the orbit of x satisfying

$$x_i^{\mathbf{l}} - x_N^{\mathbf{l}} = x_i - x_N + l_i\gamma \text{ mod } \delta \quad (1 \leq i < N).$$

Let $\ell \geq 1$. Applying the map Σ_N^y ℓ times to $x^{\mathbf{l}}$, that is $x^{\mathbf{l},1} = \Sigma_N^{x^{\mathbf{l}}}(x^{\mathbf{l}})$, $x^{\mathbf{l},2} = \Sigma_N^{x^{\mathbf{l}},1}(x^{\mathbf{l},1}), \ldots$, we arrive at a point $x^{\mathbf{l},\ell} = (x_1^{\mathbf{l},\ell}, \ldots, x_N^{\mathbf{l},\ell})$ satisfying

$$\begin{aligned} x_i^{\mathbf{l},\ell} - x_N^{\mathbf{l},\ell} &= x_i - x_N + l_i\gamma \text{ mod } \delta \quad 1 \leq i < N \\ x_N^{\mathbf{l},\ell} &= x_N^{\mathbf{l}} + \ell(N\alpha - 1) \text{ mod } \delta. \end{aligned}$$

4. One of Minkowski's theorems (see [5]) assures that for an irrational θ and ζ not representable in the form $\zeta = m\theta + n$ for any $n, m \in \mathbb{N}$, the inequality

$$\inf_{n \in \mathbb{N}} |q\theta - \zeta - n| < \frac{1}{4|q|}$$

holds for infinitely many $q \in \mathbb{Z}$.

Now apply this theorem to each $1 \leq k < N$ for $\zeta = \frac{x_k - x_N - z_k + z_N}{\delta}$ and $\theta = \frac{\gamma}{\delta}$. For suitable small perturbations of z the assumption of Minkowski's Theorem are satisfied, hence there is $q_k > \frac{\delta}{2\epsilon}$ satisfying

$$\left| q_k \frac{\gamma}{\delta} - \frac{x_k - x_N - z_k + z_N}{\delta} \right| < \frac{1}{4q_k} \text{ mod } \delta,$$

so

$$|x_k - x_N + q_k\gamma - z_k + z_N| < \frac{\epsilon}{2}.$$

Likewise we can apply the theorem to $\zeta = \frac{x_N^{\mathbf{l}} - z_N}{\delta}$ and $\theta = \frac{n\alpha - 1 \text{ mod } \delta}{\delta}$ to find q with

$$|x_N^{\mathbf{l}} + q(N\alpha - 1) - z_N| < \frac{\epsilon}{2} \quad \text{mod } \delta.$$

This implies that we may choose $\mathbf{l}$ and ℓ in $\mathbf{3}$ to find a point $\widetilde{x}$ in the orbit of x satisfying

$$\begin{aligned}
&|\widetilde{x}_k - z_k| \leq |\widetilde{x}_k - \widetilde{x}_N - z_k + z_N| + |\widetilde{x}_N - z_N| < \epsilon \quad 1 \leq k < N \\
&|\widetilde{x}_N - z_N| < \frac{\epsilon}{2} \\
&\widetilde{x}_k \in [Na - \delta, N\alpha) \quad 1 \leq k < N \\
&\widetilde{x}_N \in [N\alpha, N\alpha + \delta).
\end{aligned}$$

□

Proof of Theorem 4.2. The proof is canonical. Let $[c_1, \ldots, c_n] \times B(z_k, \frac{1}{m})$, $(c_1, \ldots, c_n \in \mathcal{A}, n, k, m \geq 1)$, be a countable family of open sets in $\mathbf{A} \times C$, where $\{z_k : k \geq 1\}$ is a dense set of points in C and where $[c_1, \ldots, c_n] = \{\mathbf{a} \in \mathbf{A} : a_i = c_i, i = 1, \ldots, n\}$. Enumerate this family as U_j, $j \geq 1$. Fix $x \in C$. Let $U_1 = [c_1, ...c_n] \times B(z_k, \frac{1}{m})$. By Proposition 4.1 there is a string $b_1, \ldots, b_q$ such that for any $\mathbf{a} \in \mathbf{A}$

$$\pi_C(T^q((b_1, \ldots, b_q, a_1, a_2, \ldots), x)) \in B\left(z_k, \frac{1}{m}\right)$$

hence

$$T^q((b_1, \ldots, b_q, c_1, \ldots, c_n, a_1, a_2, \ldots), x) \in [c_1, \ldots, c_n] \times B\left(z_k, \frac{1}{m}\right) = U_1.$$

Repeating the argument replacing x by $T^{q+n}(x)$ and U_1 by U_2 and continuing in this way we arrive at an infinite string $\mathbf{a} \in \mathbf{A}$ such that the point $(\mathbf{a}, x)$ has a dense orbit. In fact, by choosing the first string $b_1, \ldots, b_q$ in some arbitrary fixed cylinder we also see that there is a dense set of points $\mathbf{a} \in \mathbf{A}$ such that for any $x \in C$ the orbit of $(\mathbf{a}, x)$ is dense. □

Remark 4.2. Because of its applications to neural dynamics, further properties of the skew product need to be studied. First of all, ergodicity of the skew product needs to be proven. This may depend on the measure μ on the shiftspace where the measure lives. For applications it means to study different network connections of neurons. In particular, Theorem 4.2 is only proven in case that $\mathbf{A}$ is the full shiftspace, its extension to subshifts is open. The ergodic theorem is insufficient for practical purposes. Further stochastic laws for observables have to be proven, like the central limit theorem. Once this is established the stochastic laws of the observables $\xi(a, \mathbf{x})$ and $D(a, \mathbf{x})$ should be formulated. For the avalanche size ξ this has been done in [8], see also its announcement in [10]. Does ξ always have a power law with exponent $3/2$, or more cautiously, for which parameters δ, α and N does such a law hold?

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